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Integrable systems of quartic oscillators in ordinary (three-dimensional) space

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Abstract

A class of *completely integrable*, and indeed *solvable*, Hamiltonian many-body problems are exhibited, characterized by rotation-invariant Newtonian equations of motion ('acceleration equals force'), with linear and cubic forces, in ordinary (three-dimensional) space. The corresponding Hamiltonians are of normal type, with the kinetic energy *quadratic* in the canonical momenta and the potential energy *quadratic* and *quartic* in the canonical coordinates.

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1. Introduction

Recently it has been noted [1–3] that the matrix evolution equation

$$\ddot{M} = AM + MA + cM^3 \tag{1.1}$$

is *integrable*, and indeed *solvable*. Here $M \equiv M(t)$ is a square matrix of arbitrary rank, A is an arbitrary constant matrix (also square and with the same rank), c is an arbitrary 'coupling constant' (which might, of course, be rescaled away) and dots denote differentiations with respect to the independent variable t ('time'). The *integrability*, and indeed *solvability*, of this matrix evolution equation is demonstrated by recognizing [1–3] that it can be reduced to (a special case of) the non-Abelian Toda lattice, the *integrability*, and indeed *solvability* (in terms of hyperelliptic functions), of which was demonstrated some years ago by Krichever [4]. Recently it was also pointed out [1, 3, 5] that, via appropriate parametrizations of the matrix M in terms of S-dimensional vectors, the matrix evolution equation (1.1) can be reformulated as a set of coupled evolution second-order ODEs for S-vectors (and possibly, in addition, for some scalars) which have the property of being rotation-invariant (indeed covariant) in

S-dimensional space and are hence interpretable as the Newtonian equations of motion of many-body problems characterized by linear and cubic forces. Moreover, since (1.1) can be obtained from the Hamiltonian

$$H = \mathrm{trace}[\tfrac{1}{2} P^2 - MAM - (c/4)\, M^4], \tag{1.2}$$

where the (nm)-element P_{nm} of the matrix $P \equiv P(t)$ is the canonically conjugated momentum associated with the canonical variable M_{mn} (which is itself the (mn)-element of the matrix M; note the exchange of the row–column indices!), all the Newtonian equations of motion obtained in this manner are in fact obtainable from standard Hamiltonians which are the sum of a 'kinetic' part quadratic in the momenta and a 'potential' part quadratic and quartic in the coordinates.

Such quadratic and quartic oscillators are of great theoretical and applicative interest, hence a search for such many-body models which are *integrable* has always attracted much interest (see, for instance, [6] and the papers quoted therein). Of particular interest are such models in *ordinary* (three-dimensional) space with *rotation-invariant* equations of motion. Indeed much attention has recently been focused on the derivation from (1.1) of just such models via appropriate parametrizations of the matrix M in terms of *3-vectors* (and possibly also of *scalars*), parametrizations which have the property of being compatible with the evolution equation (1.1) and transforming it into a system of covariant scalar and 3-vector coupled second-order ODEs of Newtonian type [1, 3, 5].

The purpose and scope of this paper is to add to this body of knowledge by introducing an additional convenient parametrization of the matrix M, based on the algebra of the four anticommuting γ-matrices of rank 4 familiar from relativistic quantum mechanics.

In section 2 the matrix evolution equation (1.1) is rewritten in a form which is convenient for treating many-body models, and some basic properties of the representation of (4×4)-matrices in terms of 3-vectors and scalars, introduced via the four anticommuting γ matrices and their products, are reported. Our main results are then derived and displayed in section 3. Some final remarks are given in section 4.

2. Parametrization of (4 × 4)-matrices via 3-vectors and scalars

As a preliminary to using the matrix evolution equation (1.1) to generate evolution equations for several 3-vectors and scalars we rewrite it as follows:

$$\ddot{M}^{(nm)} = \sum_{l=1}^{N} [A^{(nl)}\, M^{(lm)} + M^{(nl)}\, A^{(lm)}] + c \sum_{l,k=1}^{N} M^{(nl)}\, M^{(lk)}\, M^{(km)}. \tag{2.1}$$

Here and below the indices n, m take the values $1, 2, \ldots, N$, and the N^2 evolving matrices $M^{(nm)} \equiv M^{(nm)}(t)$, as well as the N^2 constant matrices $A^{(nm)}$, are all (4×4)-matrices. Obviously this system of N^2 coupled matrix evolution equations is completely equivalent to (1.1) via the following 'block' representation of the $((4N) \times (4N))$-matrices M and A:

$$M = \begin{pmatrix} M^{(11)} & \ldots & M^{(1N)} \\ \vdots & \ddots & \vdots \\ M^{(N1)} & \ldots & M^{(NN)} \end{pmatrix}, \qquad A = \begin{pmatrix} A^{(11)} & \ldots & A^{(1N)} \\ \vdots & \ddots & \vdots \\ A^{(N1)} & \ldots & A^{(NN)} \end{pmatrix}. \tag{2.2}$$

Then let the four (4×4)-matrices γ_μ, $\mu = 0, 1, 2, 3$ be characterized by the anticommutation rules

$$\gamma_\mu\, \gamma_\nu + \gamma_\nu\, \gamma_\mu = 2\, I\, \delta_{\mu\nu}. \tag{2.3}$$

Here and in what follows Greek indices such as μ, ν take the four values 0, 1, 2, 3 unless otherwise indicated, and I is the *unit* (4×4)-matrix. It is then well known (and easily verified) that a convenient basis for (4×4)-matrices is provided by the following set of 16 (4×4)-matrices:

$$\begin{aligned} &\Gamma^S = I, \qquad \Gamma^P = \gamma_5, \qquad \Gamma^V_\mu = \gamma_\mu, \\ &\Gamma^A_\mu = \gamma_5\,\gamma_\mu, \qquad \Gamma^T_{\mu\nu} = \sigma_{\mu\nu}, \qquad \mu, \nu = 0, 1, 2, 3. \end{aligned} \tag{2.4a}$$

The γ_5 and $\sigma_{\mu\nu}$ matrices are defined in terms of the four matrices γ_μ as follows:

$$\gamma_5 = \gamma_0\,\gamma_1\,\gamma_2\,\gamma_3, \tag{2.4b}$$

$$\sigma_{\mu\nu} = \tfrac{1}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu), \qquad \mu \neq \nu, \quad \mu, \nu = 0, 1, 2, 3. \tag{2.4c}$$

The standard characterization of these 16 matrices in the context of relativistic quantum mechanics associates them with a *scalar* (Γ^S), a *pseudoscalar* (Γ^P), a 4-*vector* (Γ^V_μ), an *axial* 4-*vector* (Γ^A_μ) and an *antisymmetric tensor* ($\Gamma^T_{\mu\nu}$), accounting respectively, for *one, one, four, four* and *six* components $(1 + 1 + 4 + 4 + 6 = 16)$. In the context of interest to us, the 16 basic matrices are rather associated with four *scalars* and four *3-vectors* $(4 + 4 \times 3 = 4 + 12 = 16)$, corresponding to the following parametrization of a generic (4×4)-matrix M in terms of the four *scalars* $\rho^{(j)}$ and the four *3-vectors* $\vec{r}^{(j)} \equiv (x^{(j)},\ y^{(j)},\ z^{(j)})$:

$$M = \sum_{j=1}^{4} \rho^{(j)}\,E^{(j)} + \sum_{j=1}^{4} \vec{r}^{(j)} \cdot \vec{E}^{(j)}. \tag{2.5}$$

Here the 16 (4×4)-matrices of type E are defined as follows in terms of the original four γ-matrices:

$$\begin{aligned} &E^{(1)} = \gamma_5, \qquad E^{(2)} = I, \qquad E^{(3)} = \gamma_5\gamma_0, \qquad E^{(4)} = \gamma_0, \\ &\vec{E}^{(1)} = (\gamma_5\gamma_1, \gamma_5\gamma_2, \gamma_5\gamma_3), \qquad \vec{E}^{(2)} = (\gamma_1, \gamma_2, \gamma_3), \\ &\vec{E}^{(3)} = (\gamma_2\gamma_3, \gamma_3\gamma_1, \gamma_1\gamma_2), \qquad \vec{E}^{(4)} = (\gamma_0\gamma_1, \gamma_0\gamma_2, \gamma_0\gamma_3). \end{aligned} \tag{2.6}$$

Here and in what follows a dot sandwiched between two *3-vectors* denotes the standard *scalar product*, while a wedge will denote the standard *vector product*, for instance

$$\vec{r}^{(1)} \cdot \vec{r}^{(2)} \equiv x^{(1)}x^{(2)} + y^{(1)}y^{(2)} + z^{(1)}z^{(2)}, \tag{2.7a}$$

$$\vec{r}^{(1)} \wedge \vec{r}^{(2)} \equiv (y^{(1)}z^{(2)} - z^{(1)}y^{(2)},\ z^{(1)}x^{(2)} - x^{(1)}z^{(2)},\ x^{(1)}y^{(2)} - y^{(1)}x^{(2)}). \tag{2.7b}$$

For our purposes the crucial property of the parametrization (2.5) is the fact that it is preserved under (matrix) multiplication, in the following sense: if the two (4×4)-matrices $M^{(A)},\ M^{(B)}$ are parametrized according to (2.5),

$$M^{(C)} = \sum_{j=1}^{4} \rho^{(C)(j)}\,E^{(j)} + \sum_{j=1}^{4} \vec{r}^{(C)(j)} \cdot \vec{E}^{(j)}, \qquad C = A, B, \tag{2.8}$$

then their (matrix) product,

$$M^{(AB)} = M^{(A)}\,M^{(B)}, \tag{2.9}$$

features (of course) a similar parametrization,

$$M^{(AB)} = \sum_{j=1}^{4} \rho^{(AB)(j)}\,E^{(j)} + \sum_{j=1}^{4} \vec{r}^{(AB)(j)} \cdot \vec{E}^{(j)}, \tag{2.10}$$

with the four *scalars* $\rho^{(AB)(j)}$ and the four *3-vectors* $\vec{r}^{(AB)(j)} \equiv (x^{(AB)(j)},\ y^{(AB)(j)},\ z^{(AB)(j)})$ given in terms of the eight *scalars* $\rho^{(C)\,(k)}$ and the eight *3-vectors* $\vec{r}^{(C)\,(k)} \equiv (x^{(C)\,(k)},\ y^{(C)\,(k)},\ z^{(C)\,(k)})$, $C = A, B,\quad k = 1, 2, 3, 4$, by the following *covariant* formulae:

$$\rho^{(AB)(j)} = \sum_{k,l=1}^{4} s_{kl}^{(j)}\, \rho^{(A)(k)}\, \rho^{(B)(l)} + \sum_{k,l=1}^{4} \hat{s}_{kl}^{(j)}\, \vec{r}^{(A)(k)} \cdot \vec{r}^{(B)(l)}, \qquad j = 1, 2, 3, 4, \tag{2.11a}$$

$$\vec{r}^{(AB)(j)} = \sum_{k,l=1}^{4} u_{kl}^{(j)}\, \rho^{(A)(k)}\, \vec{r}^{(B)(l)} + \sum_{k,l=1}^{4} \hat{u}_{kl}^{(j)}\, \vec{r}^{(A)(k)}\, \rho^{(B)(l)} + \sum_{k,l=1}^{4} w_{kl}^{(j)}\, \vec{r}^{(A)(k)} \wedge \vec{r}^{(B)(l)}, \qquad j = 1, 2, 3, 4. \tag{2.11b}$$

By $s_{kl}^{(j)}, \hat{s}_{kl}^{(j)}, u_{kl}^{(j)}, \hat{u}_{kl}^{(j)}, w_{kl}^{(j)}$, $j = 1, 2, 3, 4$, we mean the (kl)-element (kth row, lth column) of the following 20 specific (4×4)-matrices:

$$s^{(1)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \qquad s^{(2)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$
$$s^{(3)} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \qquad s^{(4)} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{2.12a}$$

$$\hat{s}^{(1)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad \hat{s}^{(2)} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$
$$\hat{s}^{(3)} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad \hat{s}^{(4)} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{2.12b}$$

$$u^{(1)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad u^{(2)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$
$$u^{(3)} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad u^{(4)} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{2.12c}$$

$$\hat{u}^{(1)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \qquad \hat{u}^{(2)} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$
$$\hat{u}^{(3)} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \qquad \hat{u}^{(4)} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{2.12d}$$

$$w^{(1)} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad w^{(2)} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$
$$w^{(3)} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad w^{(4)} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{2.12e}$$

These 20 matrices could be written in more compact form in various ways, utilizing their simple block structure (half of them are block-diagonal, half of them block-antidiagonal, with the (2×2)-matrices that constitute them being themselves diagonal or antidiagonal); but we believe it is more transparent to exhibit them in fully explicit form. Note that they are all rather sparse, each of them having 12 vanishing elements and only four nonvanishing ones, equal to positive or negative unity. They are, of course, not independent, for instance $s^{(3)} = -\hat{s}^{(3)}$, $u^{(3)} = -\hat{u}^{(3)}$, $w^{(4)} = -\hat{s}^{(1)}$ and $\hat{s}^{(j)} = w^{(5-j)}$, $j = 2, 3, 4$. All the 12 matrices of type s, $\hat{s}$, w have determinant -1, while all the eight matrices of type u, $\hat{u}$ have determinant $+1$.

Let us re-emphasize that the crucial property here is the *covariant* character of formulae (2.11), entailing their *rotation-invariance*. It is because of this property that the use of this parametrization transforms the matrix evolution equation (2.1) into a *rotation-invariant* system of evolution equations of Newtonian type for *scalars* and *3-vectors*, as displayed in the following section.

Finally let us note that, in the above discussion, we have not made any distinction between *scalar* and *pseudoscalar* quantities, nor between *vectors* and *pseudovectors* (namely, between *ordinary* and *axial* 3-vectors)—namely, no distinction between quantities that behave in the same manner under rotation in three-dimensional space, but behave differently under reflections (or parity transformations). It is clear that, if such a distinction were instead introduced, one would conclude that $\rho^{(2)}$ and $\rho^{(4)}$ are *scalars*, while$\rho^{(1)}$ and $\rho^{(3)}$ are *pseudoscalars*, and likewise that $\vec{r}^{(2)}$ and $\vec{r}^{(4)}$ are *vectors* while $\vec{r}^{(1)}$ and $\vec{r}^{(3)}$ are *pseudovectors* (or, equivalently, *axial vectors*). And it is easy to verify, using the explicit form of the matrices (2.12), that this distinction is also preserved by formulae (2.11), namely that these formulae are invariant not only under (three-dimensional) rotations, but also under the parity (or space inversion) transformation.

3. Results

In this section we display the main results of this paper, which are an immediate consequence of the application of the parametrization introduced in section 2 (see (2.5), and especially (2.11)) to the matrix evolution equations (2.1). Hence these equations obtain from (2.1) by introducing the parametrization (2.5) for the (evolving) (4×4)-matrices $M^{(nm)} \equiv M^{(nm)}(t)$,

$$M^{(nm)}(t) = \sum_{j=1}^{4} \rho^{(nm)\,(j)}(t)\, E^{(j)} + \sum_{j=1}^{4} \vec{r}^{(nm)\,(j)}(t) \cdot \vec{E}^{(j)}, \tag{3.1a}$$

of course with an analogous parametrization for the constant (4×4)-matrices $A^{(nm)}$,

$$A^{(nm)} = \sum_{j=1}^{4} \alpha^{(nm)\,(j)}\, E^{(j)} + \sum_{j=1}^{4} \vec{a}^{(nm)\,(j)} \cdot \vec{E}^{(j)}. \tag{3.1b}$$

We thereby obtain the following Newtonian equations of motion:

$$\ddot{\rho}^{(nm)(j)} = \sum_{l=1}^{N}\sum_{h,k=1}^{4}[s_{hk}^{(j)}(\alpha^{(nl)(h)}\rho^{(lm)(k)} + \rho^{(nl)(h)}\alpha^{(lm)(k)})$$
$$+ \hat{s}_{hk}^{(j)}(\vec{a}^{(nl)(h)}\cdot\vec{r}^{(lm)(k)} + \vec{r}^{(nl)(h)}\cdot\vec{a}^{(lm)(k)})]$$
$$+ c\sum_{l,p=1}^{N}\sum_{h,k=1}^{4}\sum_{q,v=1}^{4}[s_{hk}^{(j)}(s_{qv}^{(h)}\rho^{(nl)(q)}\rho^{(lp)(v)} + \hat{s}_{qv}^{(h)}\vec{r}^{(nl)(q)}\cdot\vec{r}^{(lp)(v)})\rho^{(pm)(k)}$$
$$+ \hat{s}_{hk}^{(j)}(u_{qv}^{(h)}\rho^{(nl)(q)}\vec{r}^{(lp)(v)} + \hat{u}_{qv}^{(h)}\vec{r}^{(nl)(q)}\rho^{(lp)(v)}$$
$$+ w_{qv}^{(h)}\vec{r}^{(nl)(q)}\wedge\vec{r}^{(lp)(v)})\cdot\vec{r}^{(pm)(k)}], \tag{3.2a}$$

$$\ddot{\vec{r}}^{(nm)(j)} = \sum_{l=1}^{N}\sum_{h,k=1}^{4}[u_{hk}^{(j)}(\alpha^{(nl)(h)}\vec{r}^{(lm)(k)} + \rho^{(nl)(h)}\vec{a}^{(lm)(k)}) + \hat{u}_{hk}^{(j)}(\vec{a}^{(nl)(h)}\rho^{(lm)(k)}$$
$$+ \vec{r}^{(nl)(h)}\alpha^{(lm)(k)}) + w_{hk}^{(j)}(\vec{a}^{(nl)(h)}\wedge\vec{r}^{(lm)(k)} + \vec{r}^{(nl)(h)}\wedge\vec{a}^{(lm)(k)})]$$
$$+ c\sum_{l,p=1}^{N}\sum_{h,k=1}^{4}\sum_{q,v=1}^{4}[u_{hk}^{(j)}(s_{qv}^{(h)}\rho^{(nl)(q)}\rho^{(lp)(v)} + \hat{s}_{qv}^{(h)}\vec{r}^{(nl)(q)}\cdot\vec{r}^{(lp)(v)})\vec{r}^{(pm)(k)}$$
$$+ \hat{u}_{hk}^{(j)}(u_{qv}^{(h)}\rho^{(nl)(q)}\vec{r}^{(lp)(v)} + \hat{u}_{qv}^{(h)}\vec{r}^{(nl)(q)}\rho^{(lp)(v)} + w_{qv}^{(h)}\vec{r}^{(nl)(q)}\wedge\vec{r}^{(lp)(v)})\rho^{(pm)(k)}$$
$$+ w_{hk}^{(j)}(u_{qv}^{(h)}\rho^{(nl)(q)}\vec{r}^{(lp)(v)} + \hat{u}_{qv}^{(h)}\vec{r}^{(nl)(q)}\rho^{(lp)(v)}$$
$$+ w_{qv}^{(h)}\vec{r}^{(nl)(q)}\wedge\vec{r}^{(lp)(v)})\wedge\vec{r}^{(pm)(k)}]. \tag{3.2b}$$

These Newtonian equations of motion involve the $4N^2$ scalars $\rho^{(nm)(j)} \equiv \rho^{(nm)(j)}(t)$ and the $4N^2$ 3-vectors $\vec{r}^{(nm)(j)} \equiv \vec{r}^{(nm)(j)}(t)$; they feature the $4N^2$ arbitrary scalar constants $\alpha^{(nm)(j)}$ and the $4N^2$ arbitrary constant 3-vectors $\vec{a}^{(nm)(j)}$; they are clearly covariant, hence they describe a *rotation-invariant* dynamics provided the $4N^2$ constant 3-vectors $\vec{a}^{(nm)(j)}$ (which would otherwise identify privileged directions) are *all* set to *zero*. It is also easily seen from the structure of the matrices (2.12) (see the last remark at the end of the preceding section) that for $j = 1, 3$ all (nonvanishing) terms in the right-hand side of (3.2a) are *pseudoscalars* while for $j = 2, 4$ all (nonvanishing) terms in the right-hand side of (3.2a) are *scalars* (i.e. not pseudoscalars), and likewise that for $j = 1, 3$ all (nonvanishing) terms in the right-hand side of (3.2b) are *pseudovectors* while for $j = 2, 4$ all (nonvanishing) terms in the right-hand side of (3.2b) are *vectors* (i.e. not pseudovectors), so that these Newtonian evolution equations, (3.2), preserve parity (provided, of course, the constant quantities $\alpha^{(nm)(j)}$ and $\vec{a}^{(nm)(j)}$ are assumed to behave appropriately under parity; if one insists that they are just constants, then the Newtonian equations of motion (3.2) will be invariant under both rotations and inversions iff all the constants 3-vectors $\vec{a}^{(nm)(j)}$ vanish and moreover, the constants $\alpha^{(nm)(j)}$ also all vanish for j odd, $\alpha^{(nm)(1)} = \alpha^{(nm)(3)} = 0$).

As already mentioned above, these Newtonian equations of motion (3.2) are Hamiltonian. The Hamiltonian function H that yields them via the standard Hamiltonian equations,

$$\dot{\rho}^{(nm)(j)} = \partial H/\partial\pi^{(nm)(j)}, \qquad \dot{\pi}^{(nm)(j)} = -\partial H/\partial\rho^{(nm)(j)}, \tag{3.3a}$$
$$\dot{\vec{r}}^{(nm)(j)} = \partial H/\partial\vec{p}^{(nm)(j)}, \qquad \dot{\vec{p}}^{(nm)(j)} = -\partial H/\partial\vec{r}^{(nm)(j)}, \tag{3.3b}$$

is easily obtained by inserting in (1.2) the parametrizations (2.2) and (3.1) of M and A, as well as the analogous parametrization of P:

$$P = \begin{pmatrix} P^{(11)} & \dots & P^{(1N)} \\ \vdots & \ddots & \vdots \\ P^{(N1)} & \dots & P^{(NN)} \end{pmatrix}, \tag{3.4a}$$

$$P^{(nm)}(t) = \sum_{j=1}^{4} \pi^{(nm)(j)}(t)\, E^{(j)} + \sum_{j=1}^{4} \vec{p}^{(nm)(j)}(t) \cdot \vec{E}^{(j)} \tag{3.4b}$$

(and by dividing by 4, to compensate for the fact that trace $[I] = 4$ when I is the 4×4 unit matrix). It reads

$$\begin{aligned}
H = \sum_{j=1}^{N} \Bigg\{ & \frac{1}{2} \sum_{l=1}^{N} \sum_{h,k=1}^{4} \left(s_{hk}^{(2)} \pi^{(jl)(h)} \pi^{(lj)(k)} + \hat{s}_{hk}^{(2)} \vec{p}^{(jl)(h)} \cdot \vec{p}^{(lj)(k)} \right) \\
& - \sum_{l,p=1}^{N} \sum_{h,k=1}^{4} \sum_{q,v=1}^{4} [s_{hk}^{(2)} (s_{qv}^{(h)} \rho^{(jl)(q)} \alpha^{(lp)(v)} + \hat{s}_{qv}^{(h)} \vec{r}^{(jl)(q)} \cdot \vec{a}^{(lp)(v)}) \rho^{(pj)(k)} \\
& + \hat{s}_{hk}^{(2)} (u_{qv}^{(h)} \rho^{(jl)(q)} \vec{a}^{(lp)(v)} + \hat{u}_{qv}^{(h)} \vec{r}^{(jl)(q)} \alpha^{(lp)(v)} + w_{qv}^{(h)} \vec{r}^{(jl)(q)} \wedge \vec{a}^{(lp)(v)}) \cdot \vec{r}^{(pj)(k)}] \\
& - \frac{c}{4} \sum_{l,p,i=1}^{N} \sum_{h,k=1}^{4} \Bigg\{ s_{hk}^{(2)} \sum_{m,n=1}^{4} \sum_{q,v=1}^{4} [s_{mn}^{(h)} (s_{qv}^{(m)} \rho^{(jl)(q)} \rho^{(lp)(v)} \\
& + \hat{s}_{qv}^{(m)} \vec{r}^{(jl)(q)} \cdot \vec{r}^{(lp)(v)}) \rho^{(pi)(n)} + \hat{s}_{mn}^{(h)} (u_{qv}^{(m)} \rho^{(jl)(q)} \vec{r}^{(lp)(v)} \\
& + \hat{u}_{qv}^{(m)} \vec{r}^{(jl)(q)} \rho^{(lp)(v)} + w_{qv}^{(m)} \vec{r}^{(jl)(q)} \wedge \vec{r}^{(lp)(v)}) \cdot \vec{r}^{(pi)(n)}] \rho^{(ij)(k)} \\
& + \hat{s}_{hk}^{(2)} \sum_{m,n=1}^{4} \sum_{q,v=1}^{4} [u_{mn}^{(h)} (s_{qv}^{(m)} \rho^{(jl)(q)} \rho^{(lp)(v)} + \hat{s}_{qv}^{(m)} \vec{r}^{(jl)(q)} \cdot \vec{r}^{(lp)(v)}) \cdot \vec{r}^{(pi)(n)} \\
& + \hat{u}_{mn}^{(h)} (u_{qv}^{(m)} \rho^{(jl)(q)} \vec{r}^{(lp)(v)} + \hat{u}_{qv}^{(m)} \vec{r}^{(jl)(q)} \rho^{(lp)(v)} + w_{qv}^{(m)} \vec{r}^{(jl)(q)} \wedge \vec{r}^{(lp)(v)}) \rho^{(pi)(n)} \\
& + w_{mn}^{(h)} (u_{qv}^{(m)} \rho^{(jl)(q)} \vec{r}^{(lp)(v)} + \hat{u}_{qv}^{(m)} \vec{r}^{(jl)(q)} \rho^{(lp)(v)} \\
& + w_{qv}^{(m)} \vec{r}^{(jl)(q)} \wedge \vec{r}^{(lp)(v)}) \wedge \vec{r}^{(pi)(n)}] \cdot \vec{r}^{(ij)(k)} \Bigg\} \Bigg\}.
\end{aligned} \tag{3.5}$$

4. Outlook

In this paper we have identified a Hamiltonian system of linear and cubic oscillators, the linear part of which contains many *arbitrary* coupling constants, hence it includes many subcases. The *integrability*, and indeed *solvability*, of these equations of motion is implied by the fact that this system is an appropriate reduction of the matrix evolution equation (1.1), which is itself integrable, and indeed solvable [1–4]. A detailed analysis of the behaviour of these systems of unharmonic oscillators, as well as their utilization in applicative contexts, remains an interesting task for future work. A related interesting problem is whether to the *classical* integrability, and indeed solvability, of these and analogous [5] systems, there also corresponds some special property in the *quantal* context.

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